

On the Maximal Shortest Path in a Connected Component in V2V

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Abstract

In this work, a VANET (Vehicular Ad-hoc NETwork) is considered to operate on a simple lane, without infrastructure. The arrivals of vehicles are assumed to be general with any traffic and speed assumptions. The vehicles communicate through the shortest path. In this paper, we study the probability distribution of the number of hops on the maximal shortest path in a connected component of vehicles. The general formulation is given for any assumption of road traffic. Then, it is applied to calculate the z-transform of this distribution for medium and dense networks in the Poisson case. Our model is validated with the Madrid road traces of the Universitat Politècnica de Catalunya. These results may be useful for example when evaluating diffusion protocols through the shortest path in a VANET, where not only the mean but also the other moments are needed to derive accurate results.

Keywords: VANETs, shortest path routing, performance evaluation, connected component size.

1. Introduction

Evaluating Vehicular Ad-hoc NETworks (VANET) protocols is generally complex and difficult as it requires to consider many parameters. Ideally, a VANET model should take into account at least accurate radio models, accurate road traffic models and protocol models. For this reason, most VANET protocols are evaluated by simulation. Nevertheless, a formal method to identify shortest hop performance is currently very relevant, considering the current requirements and developments for Vehicle-to-Vehicle (V2V) communication in the Intelligent Transportation Systems (ITS) domain. Models given connectivity probabilities as a function of the distance between a sender and a receiver avoid complex and long simulations. It is the same for models of the time to route the information from a source to a destination or models given the number of hops in the shortest path. For example, it is interesting to know the probability that a message emitted from a source can reach a vehicle located at a certain distance ahead of the road. If this probability

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is positive, we would like to know how much hops are necessary to reach the destination, and how long time is needed. Connectivity probabilities have been already studied (e.g. [1]) but the process of the number of hops, when following the shortest path, is not exactly known (there are bounds in [2]). The goal of this paper is to address this issue.

In this paper, we model the number of hops of the maximal shortest path in a V2V connected component, which is the shortest path between the vehicle at an end of the connected component to the vehicle at the other extremity. Modelling the maximal shortest path can be directly used in combination with the connectivity probability to evaluate the number of hops in a diffusion protocol for instance. Let us call the hop density the average number of hops in a connected component divided by the average size of the connected component. Then, assuming a path exists between a sender and a receiver located at a certain distance, it can be used to approximate the number of hops between them by multiplying the hop density by the distance separating these two nodes. Using the results we present in this paper, the hop density can be estimated. More generally, modelling the maximal shortest path allows to derive the performance of new network protocols or mechanisms. Examples of such problems where models of number of hops in a connected component are useful to evaluate the solution are [3], [4] and [5]). In [3], we proposed a LTE-assisted D2D solution for dead-ends recovery. The problem is then to calculate the delay to transmit a message from a sender to the dead-end, the delay of the LTE-assisted solution and the delay between the LTE part and the RSU.

In the next section, works related to VANETs modeling are surveyed, the analytical model formulation is presented in section 3, the explicit calculation in the case of Poisson road traffic is presented in section 4. Section 5 presents the evaluation and model validation. Section 6 is dedicated to concluding remarks and future works. Most of the demonstrations are given in the annexes.

2. Related works

Regarding the VANET modeling, when the network is sparse, the information propagation speed may depend on the vehicle speed because the network may work in a delay tolerant fashion. In [8] and [9], models were presented for delay tolerant networks or to investigate the impact of disconnections in vehicle networks on the information propagation. These are interesting results but in our present work, we consider only real-time applications. As a consequence, we consider the information is sent faster than the vehicle speed, so the information transfer is stopped if there is a disconnection.

In works like [6] or [7], the number of nodes in a connected component is given. Though these works provide insightful results on cluster sizes, they did not consider the shortest path. The connectivity in VANETs was studied as in [10], [11], and [12] where the connectivity probabilities were derived. In [13], authors proposed an analogy by which the ad hoc network connectivity is modeled by a $G/D/\infty$ queue. In [1], the authors used this analogy to derive the number of cars in a connected component of vehicles in a VANET

and what they called the connectivity distance, which is the length of a vehicle connected component. In [14], the authors investigated the effect of channel randomness on connectivity, with particular emphasis on the effect of lognormal shadowing and Rayleigh fading phenomena. The connectivity analysis had been extended to the case of a Nakagami fading channel in [15]. In [16], authors studied the minimum transmitted power necessary to ensure network connectivity in a VANET under BER requirements.

These papers provide insightful results on connectivity analysis. However, they give no detail on the number of hops in the shortest path of a vehicle connected component, which is important to estimate propagation delays. In [17], the multi-hop packet delivery was approximated through an effective bandwidth approach, with the traffic parameters approximated by average ON and OFF periods. In [18] and [2], the connectivity and the number of hops through the shortest path were exactly modeled for Poisson traffics. Unfortunately, the results were given as a recursive formula, which makes it difficult to perform the analysis as a function of network parameters. In this paper, we provide a method to calculate the number of hops through the shortest path in a connected component in VANETs for any traffic assumption. We validate it against VANET traces provided by the Universitat Politècnica de Catalunya. We illustrate this method by deriving the explicit expression of the z-transform of its probability in the case of Poisson traffic assumption. In this case, we provide a closed-form expression which is valid for medium and dense networks (i.e. for networks for which the product of the node density by the coverage radius is larger than $\ln 4$). This model can then be used to derive access delays, or path lengths.

3. Analytical model

3.1. Model and assumptions

In this paper, we study the number of hops in a connected component. At a given time, vehicles are spread on the road, separated by random distances. Considering a solid disk propagation model, which may be refined later, vehicles can communicate if the interdistance between them is less than a certain coverage radius. In this case, vehicles can constitute connected components. The goal is to model the distribution of the number of hops of the maximal shortest path in a connected component, which is the shortest path between the vehicle at an end of the connected component to the vehicle at the other extremity. Note that the maximal shortest path is not the number of hosts located between the sender and the receiver at both extremities because of the shortest path routing algorithm. Actually, we assume a routing protocol is used allowing only vehicles on the shortest path to retransmit packets. In other words, if two vehicles receive at the same time the same packet from a sender, only the furthest one under the sender's coverage forwards the packet.

We consider a vehicles traffic on a straight road. If the vehicle transmission range is sufficiently large compared to the road width, this commonly used assumption makes sense. We do not make any assumption on the arrival process nor the speeds. At a given time t , the vehicles are separated by a distance $\{I_k\}_{k>0}$. The

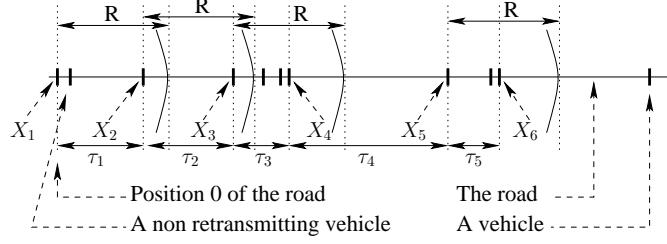


Figure 1: Example showing the sampling of the vehicles according to the shortest path

process $\{I_k\}_{k>0}$ is considered independent and identically distributed. Its cumulative distribution function is $P(I_k \leq x) = F(x)$ and its density $f(x) = \frac{dF(x)}{dx}$. Note that while the interdistances are considered independent, the traffic it models is not independent. It is so only when F corresponds to the exponential distribution. The vehicles arrive at position 0 of the road.

As in [1], to study the connectivity, we use the equivalent infinite server queueing model presented by Moriandi and Altman in [13]. These authors proved that the problem of calculating the number of wireless nodes in a connected cluster and the length of the connected path at a given time with the solid disc transmission range model is equivalent to the problem of calculating the number of clients in a busy period of the G/D/ ∞ queue and the duration of this busy period: the busy period of the G/D/ ∞ queue corresponds to a connected vehicle cluster, the time is replaced by the distance, and the communication range replaces the service time.

By using the results given by Liu and Shi in [19] on the probability generating function of the number of clients in a busy period and on the Laplace transform of the busy period duration, Yousefi et al. characterize in [1] the number of clients in a vehicle connected component and the connectivity distance, that is the length of a vehicle connected component. Here, we adopt the same approach modeling the connectivity in terms of the GP^x/D/ ∞ . Nevertheless, since we consider the number of hops on the shortest path and not each vehicle in the path and second because we do not make any assumption on the vehicle traffic, the input process of the queue is no more the Poisson process described in [1] but a sampled process of this Poisson process. As we will see in the next section, it is a Markovian process.

3.2. The interdistance process

Let $X_1, X_2, \dots, X_n, \dots$ be the **retransmitting** vehicles. X_{n+1} is the farthest vehicle under X_n 's coverage, or the next node after X_n if the distance between X_n and X_{n+1} is larger than R . Because of the shortest path routing algorithm, if X_{n+1} is the farthest vehicle under X_n coverage, it is assumed to retransmit X_n packets even if there are other vehicles between them. By extension, X_n denotes also the coordinate (location) of X_n on the road. Let τ_n be the distance between node X_n and node X_{n+1} . Let $I_{n,k}$ the k^{th} interdistance after the n^{th} vehicle. First of all, we need to characterize the process of the distance τ_n separating to retransmitting nodes.

Theorem 1. When $R - x_{n-1} \leq x_n \leq R$, the cumulative distribution function of τ_n knowing $\tau_{n-1} = x_{n-1}$ and $\tau_{n-1} \leq R$ is given by:

$$F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) = \frac{1}{1 - F(R - x_{n-1})} \int_{u=R-x_{n-1}}^{x_n} \mathcal{L}^{-1} \left[\frac{\mathcal{L}[f(x + R - x_{n-1})]}{1 - \mathcal{L}[f(x)]} \right]_{(u-(R-x_{n-1}))} [1 - F(R - u)] du \quad (1)$$

$$F_{\tau_1}(x_1) = \int_{u=0}^{x_1} \mathcal{L}^{-1} \left[\frac{\mathcal{L}[f(x)]}{1 - \mathcal{L}[f(x)]} \right]_{(u)} [1 - F(R - u)] du \quad (2)$$

where \mathcal{L} and \mathcal{L}^{-1} denotes the Laplace transform and its inverse.

When $x_n \geq R$, the cumulative distribution function of τ_n knowing $\tau_{n-1} = x_{n-1}$ and $\tau_{n-1} \leq R$ is given by:

$$F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) = P(I_{n,1} \leq x_n / I_{n,1} > R - x_{n-1}) = \frac{F(x_n) - F(R - x_{n-1})}{1 - F(R - x_{n-1})} \quad (3)$$

$$F_{\tau_1}(x_1) = F(x_1) \quad (4)$$

PROOF. See Appendix Appendix A.

We can observe from equations (1), (2), (3) and (4) that the distance between the retransmitting vehicles on the shortest path is actually a markovian process. In the following sections of the paper, we assume, without loss of generality, X_1 at the beginning of a connected component, which mean that the node X_0 before X_1 on the x-axis is out of range of x_1 : $X_1 - X_0 > R$.

In the special case of a Poisson process with rate λ ,

$$F(x) = 1 - e^{-\lambda x} \quad (5)$$

and thus,

$$\forall x_1 \leq R, F_{\tau_1}(x_1) = e^{-\lambda(R-x_1)} (1 - e^{-\lambda x_1}) \quad (6)$$

$$\forall x_1 \geq R, F_{\tau_1}(x_1) = 1 - e^{-\lambda x_1} \quad (7)$$

Also,

$$\forall x_n; R - x_{n-1} \leq x_n \leq R, F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) = e^{-\lambda(R-x_n)} - e^{-\lambda x_{n-1}} \quad (8)$$

$$\forall x_n \geq R, F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) = 1 - e^{-\lambda[x_n - (R-x_{n-1})]} \quad (9)$$

In the special case of a mixture of two exponential variables with rates λ_1 and λ_2 and weights α_1 and $\alpha_2 = 1 - \alpha_1$,

$$F(x) = 1 - \alpha_1 e^{-\lambda_1 x} - \alpha_2 e^{-\lambda_2 x} \quad (10)$$

and thus, by denoting $\lambda_m = \lambda_1(1 - \alpha_1) + \lambda_2(1 - \alpha_2)$,

$$\begin{aligned} \forall x_1 \leq R, F_{\tau_1}(x_1) &= \frac{\lambda_1 \lambda_2}{\lambda_m} \left[\alpha_1 \frac{e^{-\lambda_1 R}}{\lambda_1} (e^{\lambda_1 x} - 1) + \alpha_2 \frac{e^{-\lambda_2 R}}{\lambda_2} (e^{\lambda_2 x} - 1) \right] \\ &+ \left[\alpha_1 \lambda_1 \left(1 - \frac{\lambda_2}{\lambda_m} \right) + \alpha_2 \lambda_2 \left(1 - \frac{\lambda_1}{\lambda_m} \right) \right] \\ &\times \left[\frac{\alpha_1 e^{-\lambda_1 R}}{\lambda_m - \lambda_1} \left(1 - e^{-(\lambda_m - \lambda_1)x_1} \right) + \frac{\alpha_2 e^{-\lambda_2 R}}{\lambda_m - \lambda_2} \left(1 - e^{-(\lambda_m - \lambda_2)x_2} \right) \right] \end{aligned} \quad (11)$$

$$\forall x_1 \geq R, F_{\tau_1}(x_1) = 1 - \alpha_1 e^{-\lambda_1 x_1} - \alpha_2 e^{-\lambda_2 x_1} \quad (12)$$

Also,

$$\forall x_n; R - x_{n-1} \leq x_n \leq R,$$

$$\begin{aligned} F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) &= \alpha_1 \frac{\lambda_2}{\lambda_m} \left[e^{-\lambda_1(R-x_n)} - e^{-\lambda_1 x_{n-1}} \right] + \alpha_2 \frac{\lambda_1}{\lambda_m} \left[e^{-\lambda_2(R-x_n)} - e^{-\lambda_2 x_{n-1}} \right] \\ &+ \frac{\alpha_1 \lambda_1 \left(1 - \frac{\lambda_2}{\lambda_m} \right) e^{-\lambda_1(R-x_{n-1})} + \alpha_2 \lambda_2 \left(1 - \frac{\lambda_1}{\lambda_m} \right) e^{-\lambda_2(R-x_{n-1})}}{\alpha_1 e^{-\lambda_1(R-x_{n-1})} + \alpha_2 e^{-\lambda_2(R-x_{n-1})}} \\ &\times \left[\frac{\alpha_1 e^{-\lambda_1 R}}{\lambda_m - \lambda_1} \left(e^{-(\lambda_m - \lambda_1)(R-x_{n-1})} - e^{-(\lambda_m - \lambda_1)x} \right) \right. \\ &\left. + \frac{\alpha_2 e^{-\lambda_2 R}}{\lambda_m - \lambda_2} \left(e^{-(\lambda_m - \lambda_2)(R-x_{n-1})} - e^{-(\lambda_m - \lambda_2)x} \right) \right] e^{\lambda_m(R-x_{n-1})} \end{aligned} \quad (13)$$

$$\forall x_n \geq R, F_{\tau_n/\tau_{n-1}}(x_n, x_{n-1}) = 1 - \frac{\alpha_1 e^{-\lambda_1 x_n} + \alpha_2 e^{-\lambda_2 x_n}}{\alpha_1 e^{-\lambda_1(R-x_{n-1})} + \alpha_2 e^{-\lambda_2(R-x_{n-1})}} \quad (14)$$

3.3. Recall on the results of Liu and Shi in [19] used in [1]

Liu and Shi consider a $\text{GI}^x/\text{G}/\infty$ queue whose customers arrive in batches with generating function $A(z)$, separated by interarrival times τ_n with distribution function $F(x)$ and service time distribution $H(x)$. Y_n is the number of clients in the n^{th} batch and T_n is the time necessary to complete the services of the n^{th} batch. $K(x) = A(H(x))$ is the distribution function of T_n . In this case, they show that the z-transform of the distribution function of the number N_b of clients in a connected component is

$$\sum_{k=1}^{+\infty} P(N_b = k) z^k = \sum_{n=1}^{\infty} [A(z)]^n \int_0^{\infty} \dots \int_0^{\infty} \prod_{i=1}^{n-1} \left\{ \left[K \left(\sum_{j=i}^n x_j \right) - K(x_i) \right] dF(x_i) \right\} K(x_n) dF(x_n) \quad (15)$$

This is true because the probability distribution of the number of customers served in a busy period is

$$P(N_b = k) = \sum_{n=1}^{\infty} P \left(\sum_{i=1}^n Y_i = k \right) \times P \left(\tau_i \leq T_i < \sum_{j=i}^n \tau_j, (i = 1, \dots, n-1), \tau_n > T_n \right) \quad (16)$$

Actually, taking the generating function of (16) and noting the discrete convolutions involved in the generating function, (15) can be obtained.

Also, in the special case of the $\text{GI}^\times/\text{D}/\infty$, $K(x) = 1$ if $x \geq R$ and 0 otherwise so that (15) becomes

$$\sum_{k=1}^{+\infty} P(N_b = k) z^k = \sum_{n=1}^{\infty} [A(z)]^n \int_{x_1=0}^R \cdots \int_{x_{n-1}=0}^R \int_{x_n=R}^{\infty} \prod_{i=1}^n dF(x_i) \quad (17)$$

and thus Liu and Shi prove that formula (15) simplifies into:

$$\sum_{k=1}^{+\infty} P(N_b = k) z^k = z \cdot \frac{1 - F(R)}{1 - zF(R)} \quad (18)$$

With the same kind of arguments, Liu and Shi present the Laplace transform of the busy period for the $\text{GI}^\times/\text{G}/\infty$ which reduces, in the special case of the $\text{GI}^\times/\text{D}/\infty$, to

$$\mathcal{L}(f_d)(s) = \frac{e^{-sR} (1 - F(R))}{1 - \int_0^R e^{-sx} dF(x)} \quad (19)$$

The authors of [1] rely on (18) and (19) to derive an upper bound for the tail distribution of the number of hops in the connected component:

$$P(N_b \geq h) \leq (1 - e^{-\lambda R})^h \quad (20)$$

and the average distance of a vehicles connected component L_{cc} , which they call connectivity distance of vehicles and which is equal to the average length of the busy period:

$$\mathbb{E}[L_{cc}] = \frac{1 - e^{-\lambda R}}{\lambda e^{-\lambda R}}. \quad (21)$$

3.4. Number of hops in the maximal shortest path

Due to the independance of the interarrivals of the $\text{GI}^\times/\text{G}/\infty$, the multiple integral in (17) can be easily simplified to obtain (18). However, in our case, considering the shortest path leads to a markovian process as already explained in §3.3.2.

The process of the interdistance between the vehicles is thus not an independent process but a markovian one. As a consequence, to compute the distribution of the number of vehicles in a connected component, the results of [1] cannot be used. The authors of [1] rely on the results in [19] for the $\text{GI}^\times/\text{G}/\infty$, where the arrivals are assumed independent. Consequently, to compute the distribution of the number of vehicles in a connected component, the results given in [1] is not compatible with the new process description. That is why we roll back to the approach initiated in [1] to adapt the model according to the Markovian arrival process. Therefore, the results presented in [19] cannot be directly applied but they must be adapted to the non independent case.

Theorem 2. *In the case of a markovian interdistance process, and still denoting by $F_{\tau_i/\tau_{i-1}}(x_i, x_{i-1})$ the cumulative distribution function of τ_i knowing τ_{i-1} , the probability to have k hops in a connected component*

is:

$$\begin{aligned}
P(N_b = k) &= \int_{x_1=0}^R \int_{x_2=R-x_1}^R \cdots \int_{x_{k-1}=R-x_{k-2}}^R (1 - F_{\tau_k/\tau_{k-1}}(R, x_{k-1})) dF_{\tau_1}(x_1) \prod_{i=2}^{k-1} dF_{\tau_i/\tau_{i-1}}(x_i, x_{i-1})
\end{aligned} \tag{22}$$

PROOF. By following the same approach as in [19], but using the Markov property, we obtain:

$$\begin{aligned}
P(N_b = k) &= P\left(\forall i \in \llbracket 1; k-1 \rrbracket, \tau_i \leq R < \sum_{j=i}^k \tau_j, \tau_k > R\right) \\
&= P(\forall i \in \llbracket 1; k-1 \rrbracket, \tau_i \leq R, \tau_k > R) \\
&= \int_{x_1=0}^R P(\forall i \in \llbracket 2; k-1 \rrbracket, \tau_i \leq R, \tau_k > R/\tau_1 = x_1) dP(\tau_1 = x_1) \\
&\vdots \\
&= \int_{x_1=0}^R \int_{x_2=R-x_1}^R \cdots \int_{x_{k-1}=R-x_{k-2}}^R P(\tau_k > R/\tau_{k-1} = x_{k-1}) \times \\
&\quad \prod_{i=2}^{k-1} dP(\tau_i = x_i/\tau_{i-1} = x_{i-1}) dP(\tau_1 = x_1)
\end{aligned} \tag{23}$$

4. Explicit calculation in the case of Poisson road traffic

In this section, we derive the expression of the probabilities $P(N_b = k)$ in the case where the road traffic is Poisson with parameter λ . The Poisson assumptions means that the vehicles are independent, which is usually not the case. Nevertheless, the arrival times are often independent, even if the behavior of the cars become not independent anymore when they are on the same road. Moreover, in free flow regime, when the arrivals are separated by time arrivals larger than 5 or 6 seconds, they may be independent. It is the case in highways for example and this is why the Poisson assumption is often considered in performance evaluations. For these situations, we give the explicit solution of the model. Let us denote $\lambda' = \lambda R$, $\rho = \lambda' e^{-\lambda'}$ and $\rho' = e^{-\lambda'}$.

In the special case of the interdistance process described in §3.2 (i.e. equations (6), (7), (8) and (9)), the probability of the number of hops in the connected component becomes with a simple change of variable:

$$P(N_b = k) = \rho^{k-1} \int_{u_1=0}^1 \int_{u_2=1-u_1}^1 \cdots \int_{u_{k-1}=1-u_{k-2}}^1 \prod_{i=1}^{k-2} e^{\lambda' u_i} \prod_{i=1}^{k-1} du_i \tag{24}$$

In order to calculate this integral, it is convenient to denote

$$\mathfrak{M}_{\alpha, k} = \rho^k \int_{u_1=0}^1 \int_{u_2=1-u_1}^1 \cdots \int_{u_k=1-u_{k-1}}^1 u_k^\alpha e^{\lambda' u_k} \prod_{i=1}^{k-1} e^{\lambda' u_i} \prod_{i=1}^k du_i \tag{25}$$

All the probabilities $P(N_b)$ are function of such integrals. We can observe that

$$P(N_b = k) = \rho \mathfrak{M}_{1,k-2}. \quad (26)$$

These probabilities can be calculated by recurrence. Actually, by integrating by parts,

$$\begin{cases} \mathfrak{M}_{0,k} = \mathfrak{M}_{0,k-1} - \rho \mathfrak{M}_{1,k-2} \\ \mathfrak{M}_{\alpha,k} = \mathfrak{M}_{0,k-1} - \frac{\alpha}{\lambda'} \mathfrak{M}_{\alpha-1,k} - \frac{\rho \mathfrak{M}_{\alpha+1,k-2}}{\alpha+1} \end{cases} \quad (27)$$

$$(28)$$

Now, for medium and dense networks, that is $\lambda R \geq \ln 4$, a closed form expression can be given for the z-transform of these probabilities. Let $Q(z)$ be the z-transform of N_b and $M_1(z)$ the z-transform of $\mathfrak{M}_{1,k}$:

$$Q(z) = \sum_{k=1}^{+\infty} P(N_b = k) z^k \quad (29)$$

$$M_1(z) = \sum_{k=1}^{+\infty} \mathfrak{M}_{1,k} z^k \quad (30)$$

Theorem 3. *In the special case where $\lambda R \geq \ln 4$, the z transform $M_1(z)$ can be expressed in a close formula:*

$$M_1(z) = \frac{h_1(z) + h_2(z) + h_3(z)}{\rho z^2 \left[1 + \sqrt{1 - 4\rho' z^2} - 2ze^{\frac{1}{2}\lambda'(\sqrt{1-4\rho' z^2}-1)} \right]} \quad (31)$$

where

$$\begin{aligned} h_1(z) &= \sqrt{1 - 4\rho' z^2} [(1 - \rho' - \rho) z^3 - (1 - \rho') z^2 - z(1 - \rho) + 2 - \rho' - \rho] \\ h_2(z) &= e^{\frac{1}{2}\lambda'(\sqrt{1-4\rho' z^2}-1)} [2\rho z^3 + 2\rho' z^2 - z - 1 + (z - 1)\sqrt{1 - 4\rho' z^2}] \\ h_3(z) &= z^3(\rho' + \rho - 1) + z^2(1 - 3\rho' - 2\rho) + z(1 - \rho) + \rho' + \rho \end{aligned}$$

PROOF. See Appendix B.

The moments of the number of hops can simply be obtained by differentiating $Q(z)$.

Theorem 4. *The z-transform of N_b is:*

$$Q(z) = \rho' z + \rho z^2 (1 + M_1(z)). \quad (32)$$

PROOF.

$$\begin{aligned} P(N_b = 1) &= P(\tau_1 > R) \\ &= e^{-\lambda R} \end{aligned} \quad (33)$$

Moreover,

$$\begin{aligned} P(N_b = 2) &= \int_{x_1=0}^R P(\tau_2 > R/\tau_1 = x_1) dP(\tau_1 = x_1) \\ &= \lambda R e^{-\lambda R} \end{aligned} \quad (34)$$

Then, by using (26), we get the result.

For example, the expectation of N_b is

$$\mathbb{E}[N_b] = \frac{e^{+\frac{\lambda'}{2}\sqrt{1-4\rho'}} - \frac{2}{1+\sqrt{1-4\rho'}}e^{-\frac{\lambda'}{2}}}{e^{\frac{\lambda'}{2}} - \frac{2}{1+\sqrt{1-4\rho'}}e^{+\frac{\lambda'}{2}\sqrt{1-4\rho'}}} \quad (35)$$

5. Simulations and validations

In order to validate the proposed model, we ran a number of simulations¹ and compared these results to the analytical results. We also compared the model with traces available on the WEB and provided by the Universitat Politècnica de Catalunya.

5.1. Comparison between simulations and the model

We compare the model and the simulations in two different cases. First of all, we consider the hyperexponential case where the distance between the vehicles follows a mixture of two exponential distributions. It corresponds to the distribution (10). Then, we focus on the Poisson case.

5.1.1. The hyperexponential case

The fig. 2 presents the comparison of the model and the simulations for the hyperexponential case. Knowing the distribution of the distance between the vehicles given by (10), we can determine the distribution of the distance between two hops in a connected component by using (1), (2), (3) and (4). The result is given in (11), (12), (13) and (14). Then, we can evaluate (22) with a Monte-Carlo method. This curve is presented on the fig. 2 for different values of α_1 , λ_1 and λ_2 (with $\alpha_2 = 1 - \alpha_1$). Note that the average of the three distributions is the same, that is the mean rate of the traffic remains the same while the parameters of the exponential mixtures are varied. As expected, the size of the connected component decreases when the traffic becomes more and more bursty. Conversely, it increases with R . The simulations and the model fit very well.

The validity of (13) and (14) is presented Fig. 3. It represents the distribution of the distance between two hops τ_n knowing τ_{n-1} , for the analytical model and for a simulation with $\tau_{n-1} = 80$, $R = 100$, $\alpha_1 = 0.95$, $\lambda_1 = 1/14.2$ and $\lambda_2 = 1/43.2$.

5.1.2. The Poisson case

We compare the results of the analytical model, and more precisely the average number of hops in a connected component given by (35), with the simulations. Figure 4 shows the average number of hops in a connected component, minus the first node, in function of different values for λ' . There is a perfect match between the model and the simulations. It is interesting to note that the expression of the average number of hops given in equation (35), though theoretically valid only for $\lambda' > \ln 4$, seems to be still valid for any λ' .

¹The corresponding code is available at <http://www-public.it-sudparis.eu/~marot/archiveForReviewers.tar.gz>

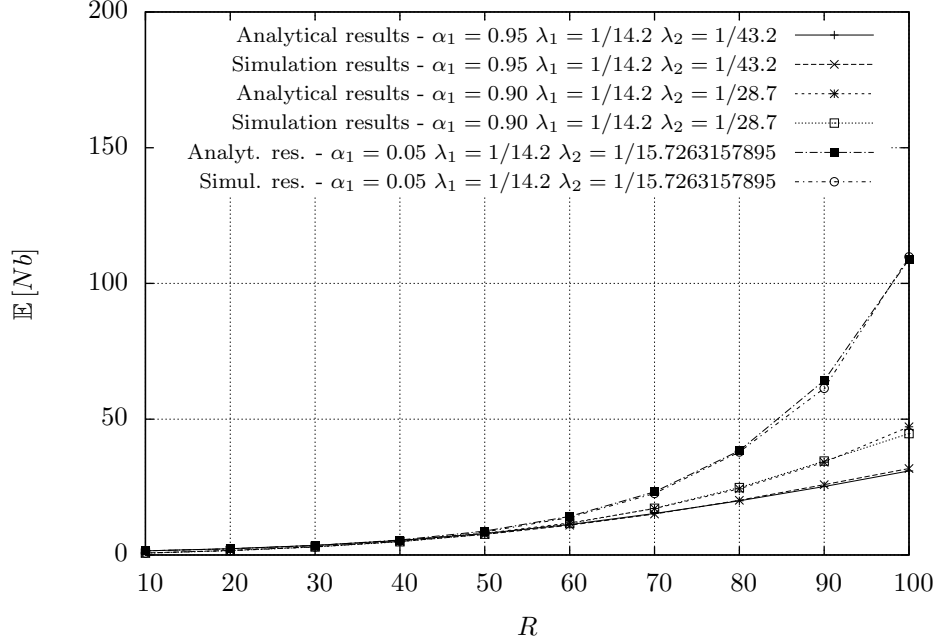


Figure 2: Average number of hops in a connected component, in function of the coverage radius R , for different values of parameters α_1 , λ_1 and λ_2 (with $\alpha_2 = 1 - \alpha_1$) at constant mean rate

Actually, we can notice that for $\lambda' \in [0 : \ln 4]$ when injecting $i\sqrt{4e^{-\lambda'} - 1}$ (where, here, $i^2 = -1$) in place of $\sqrt{1 - 4e^{-\lambda'}}$ in eq. (35), the imaginary part disappears so that, it becomes:

$$\frac{e^{+\frac{\lambda'}{2}\sqrt{1-4\rho'}} - \frac{2}{1+\sqrt{1-4\rho'}}e^{-\frac{\lambda'}{2}}}{e^{\frac{\lambda'}{2}} - \frac{2}{1+\sqrt{1-4\rho'}}e^{+\frac{\lambda'}{2}\sqrt{1-4\rho'}}} = \frac{2e^{-\frac{\lambda'}{2}} \cos \frac{\lambda'\sqrt{4e^{-\lambda'}-1}}{2} - 1}{2 - e^{\frac{\lambda'}{2}} \left[\cos \frac{\lambda'\sqrt{4e^{-\lambda'}-1}}{2} + \sqrt{4e^{-\lambda'}-1} \sin \frac{\lambda'\sqrt{4e^{-\lambda'}-1}}{2} \right]} \quad (36)$$

5.2. Comparison with traces of the Universitat Politècnica de Catalunya

To validate our results, we compared our model to vehicular traces provided by the Universitat Politècnica de Catalunya (cf. [20] and [21]). Since this traffic is not Poisson, we determined the distribution of the distance between vehicles and then we injected this distance in (1), (2), (3) and (4) in order to determine the distribution of the distance between hops. Using this estimated distribution, we calculated the mean of the number of hops in the maximal shortest path by numerically evaluating (22) with a Monte-Carlo method. With the distribution of the distances between vehicles we just estimated, we also generated artificially the same traffic and finally we compared the size of the number of hops in the maximal shortest path for the traces, for the artificially generated traffic and the model (i.e. (22)).

5.2.1. Traces analysis

As explained in [20], these traces are derived from high-detail real-world traffic counts and describe the road traffic on two highways around Madrid, Spain, at several hours of different working days. The Autovía

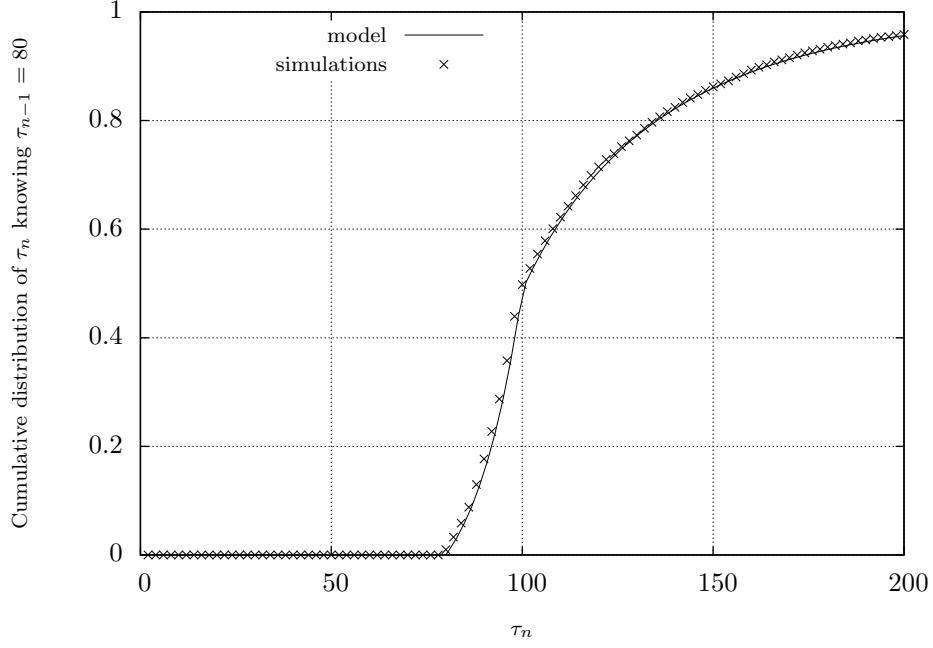


Figure 3: Distribution of τ_n knowing $\tau_{n-1} = 80$ for $R = 100$, $\alpha_1 = 0.95$, $\lambda_1 = 1/14.2$ and $\lambda_2 = 1/43.2$

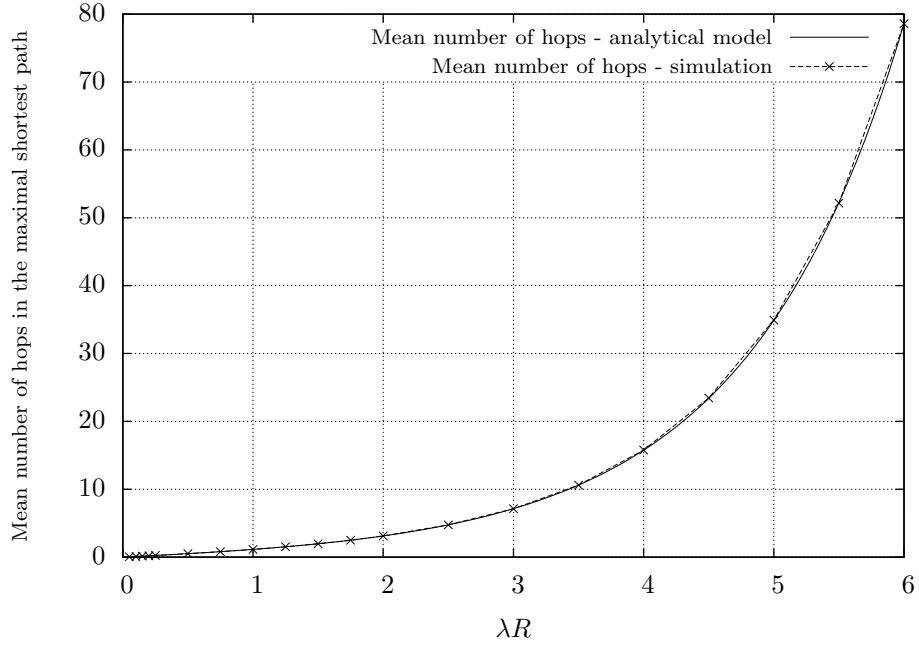


Figure 4: Average number of hops in a connected component, 1st vehicle excluded, in function of λ'

A6 is a motorway that connects the city of A Coruña to the city of Madrid. This road enters into the urban area from the northwest and collects the traffic demand of the conurbation that was built along it. The data

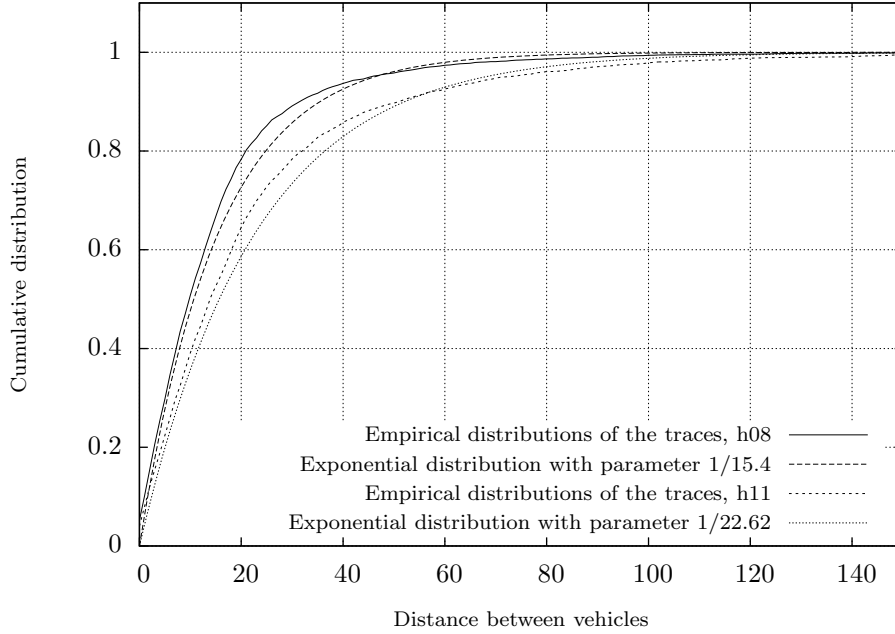


Figure 5: Cumulative distribution function of the distance (in meters) between the vehicles for the traces and for the corresponding exponential distributions with same means

collection point is placed around the 11-km milepost (Madrid direction), where the A6 features three lanes with a speed limit of 120 km/h. However, this location is right after a popular entrance ramp that joins the rightmost lane and significantly slows down the road traffic. The length of the considered road segment is 10km.

The road traffic data set is formed by mobility traces for 4 dates and 2 time intervals (8h am and 11h am). To avoid transient effects, we did not consider the beginning of the files but we kept the data only after 600 seconds of measurements. Then, since the four data sets are more or less homogeneous at a given time interval (i.e. either 8h am or 11h am), we concatenated the remaining parts of the corresponding four data sets. Also, we sampled the state of the traffic every 600s.

We approximated the empirical cumulative distributions function $F_{8h}(x)$ and $F_{11h}(x)$ of the distance between each vehicle in each sample (8h or 11h) by the following hyperexponential distributions:

$$F_{8h}(x) = 1 - 0.92e^{-x/12.1534} - 0.08e^{-x/40.4655} \quad (37)$$

$$F_{11h}(x) = 1 - 0.8100133e^{-x/16.20606} - 0.1899867e^{-x/44.4476} \quad (38)$$

On fig. 5 the cumulative distributions of the distance in meters between the vehicles for the traces is compared to the corresponding exponential distributions with same means. We observe that they differ. Then, on fig. 6, they are compared to (37) and (38). The fitting is good.

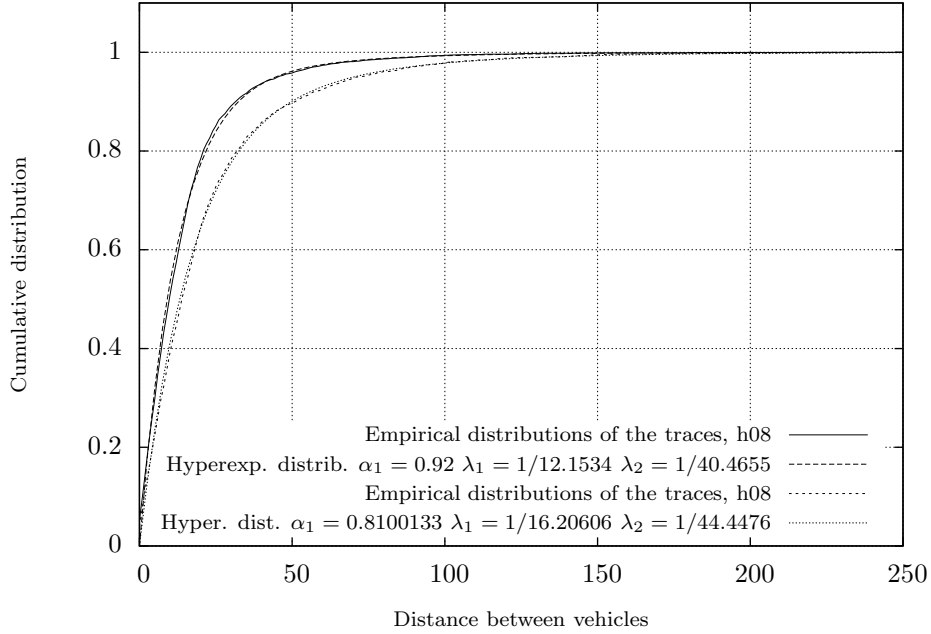


Figure 6: Cumulative distribution function of the distance (in meters) between the vehicles for the traces and for the corresponding hyperexponential distributions with fitted parameters

5.2.2. Comparison between the traces and the model

The comparison of the number of hops in the maximal shortest path (or number of hops in a connected component) for the traces, the analytical model and the simulations is given Fig. 7. The match is not perfect due to the fact that the estimation of the distribution of the distances between the vehicles is also no perfect. Actually, we decided to approximate it by hyperexponential laws as mixtures of two exponential distributions but maybe could it be more accurate to use a mixture of more exponential laws than only two. Nevertheless, the results are sufficiently close to validate our model.

5.2.3. Remark on the density of hop per unit length in a connected component

On Fig. 8, the average number of hops of the connected component divided by the length of the connected component is plotted in function of the coverage radius R for the hyperexponential distributions with various parameters α_1 , $\alpha_2 = 1 - \alpha_1$, λ_1 and λ_2 . The unit length is the radius R . To calculate the length of the connected component, we used (19), which gives, in the case of the considered hyperexponential process:

$$\mathbb{E}[L_{cc}] = \frac{1}{\lambda_1 \lambda_2} \frac{\alpha_1 \lambda_2 (1 - e^{-\lambda_1 R}) + \alpha_2 \lambda_1 (1 - e^{-\lambda_2 R})}{\alpha_1 e^{-\lambda_1 R} + \alpha_2 e^{-\lambda_2 R}} \quad (39)$$

All the curves corresponds to different parameters of the exponential mixture, but the resulting mean rate is kept constant. Thus, it can be noticed that this density is not uniform with the traffic conditions: it depends on the traffic profile.

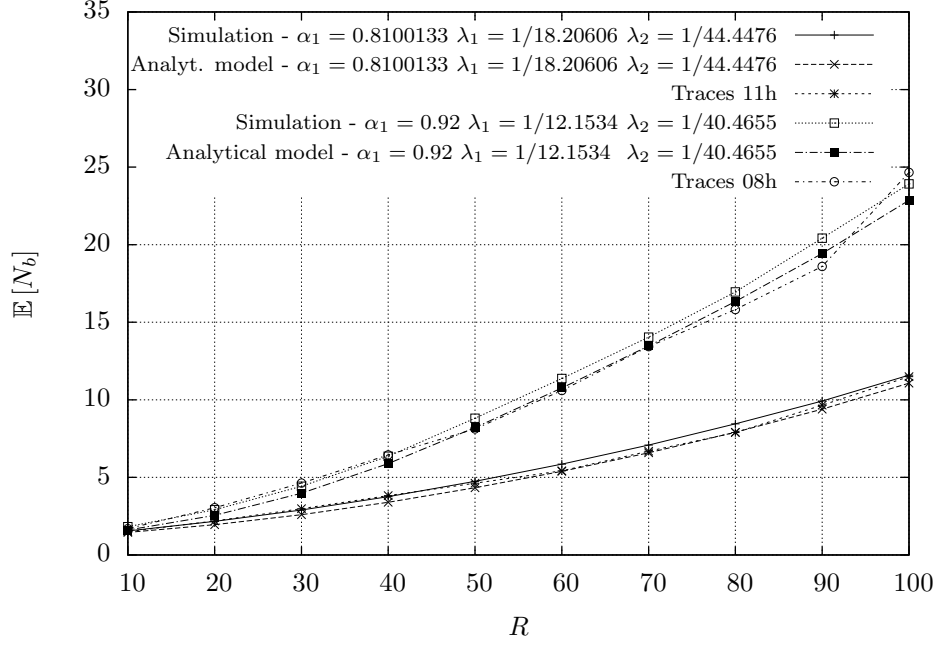


Figure 7: Average number of hops in a connected components, for the traces, the analytical model and the simulations with the corresponding hyperexponential distributions (37) and (38)

6. Conclusion

VANET modeling is a requirement to avoid complex simulations when designing new protocols. For studies involving routing mechanisms, the number of hops along the shortest path must be characterized, especially (but not only) for VANET without infrastructure. In this paper, we propose a method to calculate the distribution of the number of hops in the maximal shortest path of a vehicle connected component for any traffic assumption. Also, for the special case of the Poisson assumption, we provide a closed-form expression of its z-transform for medium and dense networks. We derive also the average of this number and it is easy to obtain the other moments by following the same method as for deriving the expectation either analytically or numerically. We validated our approach with traces available at the Universitat Politècnica de Catalunya.

These results can be exactly used to know the average number of hops for diffusion protocols in VANETs or other performance studies like in [3]. It can also be used to exactly estimate the time needed to diffuse the information by multiplying the number of hops on the shortest path by the access time at each hop since it can be assumed an independance between the access to the medium and the spread of the connected component, even if both numbers increase with the road traffic intensity. It can also be used to approximate the number of hops on a path over a certain distance separating the source and the destination by estimating the hop density of the shortest path and multiplying it by the distance, the hop density being the average

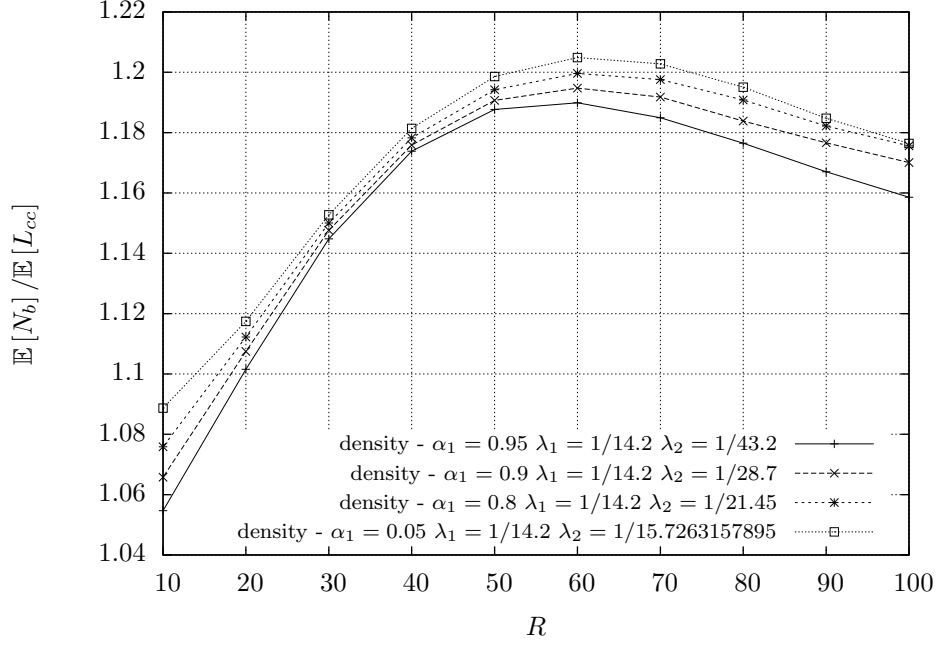


Figure 8: Average number of hops in a connected components divided by the average length of the connected component ("density of hops") in function of the coverage radius R , for the hyperexponential distributions with various parameters α_1 , $\alpha_2 = 1 - \alpha_1$, λ_1 and λ_2

given in this paper divided by the average spread of the connected component given in [1].

Appendix A. Demonstration of theorem 1

The probability that τ_n be less than x_n knowing that $\tau_{n-1} = x_{n-1}$ is:

$$\begin{aligned}
 F(\tau_n \leq x_n / \tau_{n-1} = x_{n-1}) &= P(\tau_n \leq x_n / I_{n-1,1} \geq R - x_{n-1}) \\
 &= \frac{P(\tau_n \leq x_n \cap I_{n-1,1} \geq R - x_{n-1})}{P(I_{n-1,1} \geq R - x_{n-1})} \\
 &= \frac{P(\tau_n \leq x_n \cap I_{n-1,1} \geq R - x_{n-1})}{1 - F(R - x_{n-1})}
 \end{aligned} \tag{A.1}$$

and the probability to have $\tau_n \leq x_n \cap I_{n-1,1}$ is the probability to have any number of vehicles between $X_n + R - x_{n-1}$ and x on one hand and on the other hand that the next interdistance between the last vehicle just before x and the next vehicle to be larger than $R - x_n$. Its density is:

$$\begin{aligned}
& \frac{dP}{dx_n} (\tau_n = x_n \cap I_{n-1,1} \geq R - x_{n-1}) \\
&= \frac{dP}{dx_n} (I_{n-1,1} = x_n) P(I_{n-1,2} \geq R - x_n) \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} \frac{dP(I_{n-1,1} = i_{n-1,1})}{di_{n-1,1}} \left(\frac{dP(I_{n-1,2} = i_{n-1,2})}{di_{n-1,2}} \right)_{i_{n-1,2}=x_n-i_{n-1,1}} di_{n-1,1} P(I_{n-1,3} \geq R - x_n) \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} \int_{i_{n-1,2}=0}^{x_n-i_{n-1,1}} \frac{dP(I_{n-1,1} = i_{n-1,1})}{di_{n-1,1}} \frac{dP(I_{n-1,2} = i_{n-1,2})}{di_{n-1,2}} \times \\
&\quad \left(\frac{dP(I_{n-1,3} = i_{n-1,3})}{di_{n-1,3}} \right)_{i_{n-1,3}=x_n-i_{n-1,1}-i_{n-1,2}} di_{n-1,1} di_{n-1,2} P(I_{n-1,4} \geq R - x_n) \\
&+ \dots \\
&= f(x_n) [1 - F(R - x_n)] \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} f(i_{n-1,1}) f(x_n - i_{n-1,1}) di_{n-1,1} [1 - F(R - x_n)] \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} \int_{i_{n-1,2}=0}^{x_n-i_{n-1,1}} f(i_{n-1,1}) f(i_{n-1,2}) f(x_n - i_{n-1,1} - i_{n-1,2}) di_{n-1,1} di_{n-1,2} [1 - F(R - x_n)] \\
&+ \dots \dots \dots
\end{aligned} \tag{A.2}$$

Let us define $g(x_n)$ such as:

$$\begin{aligned}
g(x_n) &= f(x_n) \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} f(i_{n-1,1}) f(x_n - i_{n-1,1}) di_{n-1,1} \\
&+ \int_{i_{n-1,1}=R-x_{n-1}}^{x_n} \int_{i_{n-1,2}=0}^{x_n-i_{n-1,1}} f(i_{n-1,1}) f(i_{n-1,2}) f(x_n - i_{n-1,1} - i_{n-1,2}) di_{n-1,1} di_{n-1,2} \\
&+ \dots \dots \dots
\end{aligned} \tag{A.3}$$

Then,

$$\frac{dP}{dx_n} (\tau_n = x_n \cap I_{n-1,1} \geq R - x_{n-1}) = g(x_n) [1 - F(R - x_n)] \tag{A.4}$$

and since for any function h

$$\int_{x=R-x_{n-1}}^{+\infty} e^{-sx} h(x) dx = e^{-s(R-x_{n-1})} \mathcal{L}[h(x + R - x_{n-1})], \tag{A.5}$$

we deduce from the definition of g that

$$\begin{aligned}
\int_{x=R-x_{n-1}}^{+\infty} e^{-sx} g(x) dx &= e^{-s(R-x_{n-1})} \mathcal{L}[g(x + R - x_{n-1})] \\
&= e^{-s(R-x_{n-1})} \mathcal{L}[f(x + R - x_{n-1})] \sum_{k=0}^{+\infty} \mathcal{L}[f(x)]^k
\end{aligned}$$

so that

$$\mathcal{L}[g(x + R - x_{n-1})] = \frac{\mathcal{L}[f(x + R - x_{n-1})]}{1 - \mathcal{L}[f(x)]} \quad (\text{A.6})$$

Combining (A.4) and (A.6), integrating on x_n between $R - x_{n-1}$ and x and using (A.1), we get (1).

Appendix B. Demonstration of theorem 3

The idea of the proof is to begin by finding the expression of $M_1(z)$, given by lemma 2, which is proved itself in Appendix C. Lemmas 3, 4, 5 and 6 are then used to simplify the expression of $M_1(z)$ given by lemma 2. The lemma 1 gives the condition of convergence of $M_1(z)$.

Let us define the following notations.

$$b_i = \rho'^i \left(\sum_{j=0}^i (-1)^j \frac{\lambda'^j}{j!} C_{2i-j}^i \right) \quad (\text{B.1})$$

$$c_i = \rho'^i C_{2i-1}^i \quad (\text{B.2})$$

$$a_{2k,i} = \rho'^{k-1} (-1)^i C_{2k-i-3}^{k-2} \quad (\text{B.3})$$

$$a_{2k+1,i} = \rho'^k (-1)^i C_{2k-i-1}^{k-1}. \quad (\text{B.4})$$

Let us also define the following z-transforms.

$$\forall i > 0, A_i^{(e)}(z) = \sum_{k=i+1}^{+\infty} a_{2k,i} z^{2k} \quad (\text{B.5})$$

$$A_0^{(e)}(z) = \sum_{k=2}^{+\infty} a_{2k,0} z^{2k} \quad (\text{B.6})$$

$$\forall i > 0, A_i^{(o)}(z) = \sum_{k=i}^{+\infty} a_{2k+1,i} z^{2k+1} \quad (\text{B.7})$$

$$A_0^{(o)}(z) = \sum_{k=1}^{+\infty} a_{2k+1,0} z^{2k+1} \quad (\text{B.8})$$

Lemma 1. *The series $\sum_{i=0}^{+\infty} b_i z^{2i+1}$ converges for $\lambda' > \ln 4 + \ln z^2$.*

PROOF.

$$\begin{aligned} b_i &= \rho'^i \left(\sum_{j=0}^i (-1)^j \frac{\lambda'^j}{j!} C_{2i-j}^i \right) \\ &= \rho'^i \left(C_{2i}^i \sum_{j=0}^i \frac{(-\lambda')^j}{j!} \frac{C_{2i-j}^i}{C_{2i}^i} \right) \\ &= \rho'^i \left(C_{2i}^i \sum_{j=0}^i \frac{(-\lambda')^j}{j!} \frac{(2i-j)! i!}{(2i)!(i-j)!} \right) \end{aligned} \quad (\text{B.9})$$

But

$$\sum_{j=0}^i \frac{(-\lambda')^j}{j!} \frac{(2i-j)!i!}{(2i)!(i-j)!}$$

is bounded because

$$\begin{aligned} \frac{(2i-j)!i!}{(2i)!(i-j)!} &= \frac{i(i-1)\cdots(i-j+1)}{2i(2i-1)\cdots(2i-j+1)} \\ &< 1. \end{aligned} \quad (\text{B.10})$$

Moreover, in the neighborhood of $+\infty$, by denoting Cst a constant term, and using Stirling's formula,

$$\begin{aligned} b_i z^{2i+1} &\sim \rho'^i Cst \times C_{2i}^i z^{2i+1} \\ &\sim \rho'^i Cst \frac{(2i)!}{i!i!} z^{2i+1} \\ &\sim \rho'^i Cst \frac{\sqrt{2\pi 2i} \left(\frac{2i}{e}\right)^{2i}}{\left[\sqrt{2\pi i} \left(\frac{i}{e}\right)^i\right]^2} z^{2i+1} \\ &\sim Cst \frac{e^{i \ln 4 - i \lambda'}}{\sqrt{\pi i}} z^{2i+1} \end{aligned} \quad (\text{B.11})$$

Then, $\sum_{i=0}^{+\infty} b_i z^{2i+1}$ converges if and only if $\lambda' > \ln 4 + 2 \ln z$.

Similarly, it can be shown that the other sums converges only when $\lambda' > \ln 4$ for z in the neighborhood of 1. Then, for $\lambda' > \ln 4$ for z in the neighborhood of 1, we have the following lemma.

Lemma 2.

$$\begin{aligned} M_1(z) &= \frac{1}{\rho z^2 \left[1 - b_0 z + \sum_{i=1}^{+\infty} (c_i z^{2i} - b_i z^{2i+1}) \right]} \times \\ &\left[(z-1) \sum_{i=0}^{+\infty} \frac{\lambda'^i}{i!} \left(\mathfrak{M}_{i,1} A_i^{(o)}(z) + \mathfrak{M}_{i,2} A_i^{(e)}(z) \right) \right. \\ &\left. + (\mathfrak{M}_{0,1} z + \mathfrak{M}_{0,2} z^2 - \mathfrak{M}_{0,1} z^2) \sum_{i=1}^{+\infty} c_i z^{2i} - \mathfrak{M}_{0,2} z^2 \sum_{i=1}^{+\infty} b_i z^{2i+1} \right] \end{aligned} \quad (\text{B.12})$$

where, for all $\alpha \geq 0$,

$$\mathfrak{M}_{\alpha,1} = \sum_{i=0}^{\alpha} \frac{(-1)^i \alpha!}{(\alpha-i)!} \frac{1}{\lambda'^i} + (-1)^{\alpha+1} \frac{\alpha!}{\lambda'^{\alpha}} \rho' \quad (\text{B.13})$$

and, for all $\alpha \geq 0$,

$$\mathfrak{M}_{\alpha,2} = \sum_{i=0}^{\alpha} \frac{(-1)^i \alpha!}{(\alpha-i)!} \frac{1}{\lambda'^i} - \frac{\rho}{\alpha+1} - (-1)^{\alpha} \frac{\alpha!}{\lambda'} \rho' \quad (\text{B.14})$$

PROOF. See Appendix Appendix C.

At last, by combining lemmas 3, 4, 5 and 6, this expression of $M_1(z)$ can be simplified into (31).

Let

$$u_{\alpha,k} = \frac{\lambda'^\alpha}{\alpha!} \mathfrak{M}_{\alpha,k}. \quad (\text{B.15})$$

Lemma 3. *If $\lambda R \geq \ln 4$,*

$$\sum_{\alpha=1}^{+\infty} u_{\alpha,1} A_\alpha^{(o)}(z) = \frac{f(z)}{1 + \frac{1}{2}g(z)} \left[\frac{1}{2}g(z)(\rho' - 1) + e^{\frac{1}{2}\lambda'g(z)} - 1 \right] \quad (\text{B.16})$$

where

$$f(z) = \frac{2z(\rho'z^2)}{\sqrt{1 - 4\rho'z^2} - (1 - 4\rho'z^2)} \quad (\text{B.17})$$

and

$$g(z) = \sqrt{1 - 4\rho'z^2} - 1 \quad (\text{B.18})$$

PROOF. From equations (B.4) and (B.7), $\forall i > 0$,

$$A_i^{(o)}(z) = \rho'^i (-1)^i z^{2i+1} \sum_{k=0}^{+\infty} \rho'^k C_{2k+i-1}^k z^{2k}. \quad (\text{B.19})$$

But,

$$\frac{\rho'^{k+1} C_{2(k+1)+i-1}^{k+1}}{\rho'^k C_{2k+i-1}^k} = 4\rho' \frac{\left[k + \frac{i+1}{2} \right] \left[k + \frac{i}{2} \right]}{(k+1)(k+i)} \quad (\text{B.20})$$

so that the series $\sum_{k=0}^{+\infty} \rho'^k C_{2k+i-1}^k z^{2k}$ is the hypergeometric series:

$$\sum_{k=0}^{+\infty} \rho'^k C_{2k+i-1}^k z^{2k} = {}_2F_1 \left(\frac{i+1}{2}, \frac{i}{2}, i, 4\rho'z^2 \right). \quad (\text{B.21})$$

Also,

$${}_2F_1 \left(a, b, 2b, \frac{4u}{(u+1)^2} \right) = (u+1)^{2a} {}_2F_1 \left(a, a-b + \frac{1}{2}, b + \frac{1}{2}, u^2 \right) \quad (\text{B.22})$$

and

$${}_2F_1(a, 1, a, u^2) = (1 - u^2)^{-1}. \quad (\text{B.23})$$

Thus, by chosing u so that

$$\frac{4u}{(u+1)^2} = 4\rho'z^2, \quad (\text{B.24})$$

we have

$$\forall i > 0, A_i^{(o)}(z) = f(z) \frac{1}{2^i} g(z)^i. \quad (\text{B.25})$$

At last,

$$\sum_{\alpha=1}^{+\infty} u_{\alpha,1} A_{\alpha}^{(o)}(z) = \sum_{\alpha=1}^{+\infty} (-1)^{\alpha} \left[\sum_{j=0}^{\alpha} (-1)^j \frac{\lambda'^j}{j!} - \rho' \right] f(z) \frac{1}{2^{\alpha}} g(z)^{\alpha} \quad (\text{B.26})$$

and noting that

$$\sum_{\alpha=1}^{+\infty} \sum_{j=0}^{\alpha} v_j w_{\alpha} = \sum_{\alpha=1}^{+\infty} v_0 w_{\alpha} + \sum_{j=1}^{+\infty} \sum_{\alpha=j}^{+\infty} v_j w_{\alpha}, \quad (\text{B.27})$$

where (v_n) and (w_n) are two arbitrary sequences, we get (B.16). Note that it exists a u solution of (B.24) only if $\lambda R \geq \ln 4$.

Similarly, by noting that $a_{2k,i} = a_{2k-1,i}$, the following lemma can be proved.

Lemma 4. *If $\lambda R \geq \ln 4$,*

$$\sum_{\alpha=1}^{+\infty} u_{\alpha,2} A_{\alpha}^{(e)}(z) = \frac{2\rho' z f(z)}{g(z)} \left[1 + \frac{1}{2} \lambda' g(z) - e^{\frac{1}{2} \lambda' g(z)} \right] + \frac{1}{2} (\rho' - 1) \frac{z f(z) g(z)}{1 + \frac{1}{2} g(z)} + z f(z) \frac{e^{\frac{1}{2} \lambda' g(z)} - 1}{1 + \frac{1}{2} g(z)} \quad (\text{B.28})$$

Lemma 5. *If $\lambda R \geq \ln 4$,*

$$\sum_{i=1}^{+\infty} c_i z^{2i} = \frac{1}{4\rho' z^3} f(z) g(z)^2. \quad (\text{B.29})$$

PROOF. On one hand,

$$\sum_{i=1}^{+\infty} c_i z^{2i} = \rho' z^2 {}_2F_1 \left(\frac{3}{2}, 1, 2, 4\rho' z^2 \right) \quad (\text{B.30})$$

and on the other hand,

$$\begin{aligned} {}_2F_1 \left(\frac{3}{2}, 1, 2, \frac{4u}{(1+u)^2} \right) &= (u+1)^3 {}_2F_1 \left(\frac{3}{2}, 1, \frac{3}{2}, u^2 \right) \\ &= \frac{(u+1)^3}{1-u^2}. \end{aligned} \quad (\text{B.31})$$

Thus, with u solution of (B.24), (B.29) is proved.

Lemma 6. *If $\lambda R \geq \ln 4$,*

$$\sum_{i=1}^{+\infty} b_i z^{2i+1} = \frac{z \left[e^{\frac{1}{2} \lambda' g(z)} - g(z) - 1 \right]}{1 + g(z)}. \quad (\text{B.32})$$

PROOF. This lemma can be proved by using identity (B.27) and the following ones:

$${}_2F_1 \left(\frac{1}{2}, 1, 1, 4\rho' z^2 \right) = \frac{1}{\sqrt{1-4\rho' z^2}} \quad (\text{B.33})$$

and

$${}_2F_1 \left(a, b, 2b, \frac{4u}{(u+1)^2} \right) = (1+u)^{2a} {}_2F_1 \left(a, a-b+\frac{1}{2}, b+\frac{1}{2}, u^2 \right) \quad (\text{B.34})$$

Appendix C. Demonstration of lemma2

In this section, we derive $M_1(z)$. To find the elements of the sequence $(\mathfrak{M}_{1,k})_k$ it is necessary to solve the double recurrence, both in α and k . That is why we will proceed in two steps. The set of the positive values of α and k constitutes a quarter plane. In the first step, we will bring back the recurrence in the borders of the quarter plane by eliminating the terms in $\mathfrak{M}_{\alpha,k}$ for which $\alpha > 0$ and $k > 2$ and keep only the terms of the form $\mathfrak{M}_{0,k}$ for $k > 2$ and the terms of the form $\mathfrak{M}_{\alpha,1}$ or $\mathfrak{M}_{\alpha,2}$. Actually, $\mathfrak{M}_{\alpha,1}$ and $\mathfrak{M}_{\alpha,2}$ are easy to calculate. This way, we will obtain a new recurrence expressing $\mathfrak{M}_{0,k}$ in function of $\mathfrak{M}_{0,i}$ where $i < k$, $\mathfrak{M}_{\alpha,1}$ and $\mathfrak{M}_{\alpha,2}$. In the second step, we will solve this last recurrence in order to find $\mathfrak{M}_{0,k}$ in function of the $\mathfrak{M}_{\alpha,1}$ and $\mathfrak{M}_{\alpha,2}$. Then, using equation (27), it is easy to deduce $\mathfrak{M}_{1,k}$.

Appendix C.1. First step - expressing $\mathfrak{M}_{0,k}$ in function of $\mathfrak{M}_{0,i}$, $i < k$, $\mathfrak{M}_{\alpha,1}$ and $\mathfrak{M}_{\alpha,2}$

By injecting (B.15) in (27) and (28), a simpler system is obtained:

$$\begin{cases} u_{0,k} = u_{0,k-1} - \rho' u_{1,k-2} \\ u_{\alpha,k} = \frac{\lambda'^\alpha}{\alpha!} u_{0,k-1} - u_{\alpha-1,k} - \rho' u_{\alpha+1,k-2} \end{cases} \quad \begin{matrix} \text{(C.1)} \\ \text{(C.2)} \end{matrix}$$

Let $V_k(z)$ be the z-transform of $u_{\alpha,k}$ relatively to α :

$$V_k(z) = \sum_{\alpha=0}^{+\infty} u_{\alpha,k} z^\alpha \quad \text{(C.3)}$$

Multiplying (C.2) by z^α and summing over $\alpha \geq 1$,

$$V_k(z) = \frac{e^{\lambda' z}}{1+z} u_{0,k-1} + \frac{\rho'}{z(1+z)} u_{0,k-2} - \frac{\rho'}{z(1+z)} V_{k-2}(z) \quad \text{(C.4)}$$

By recurrence,

$$V_{2k}(z) \quad \text{(C.5)}$$

$$= \left[-\frac{\rho'}{z(1+z)} \right]^{k-1} V_2(z) - \sum_{i=1}^{k-1} \left[-\frac{\rho'}{z(1+z)} \right]^i u_{0,2(k-i)} + \frac{e^{\lambda' z}}{1+z} \sum_{i=0}^{k-2} \left(-\frac{\rho'}{z(1+z)} \right)^i u_{0,2(k-i)-1}$$

$$V_{2k+1}(z) \quad \text{(C.6)}$$

$$= \left[-\frac{\rho'}{z(1+z)} \right]^{k-1} V_1(z) - \sum_{i=1}^k \left[-\frac{\rho'}{z(1+z)} \right]^i u_{0,2(k-i)+1} + \frac{e^{\lambda' z}}{1+z} \sum_{i=0}^{k-1} \left(-\frac{\rho'}{z(1+z)} \right)^i u_{0,2(k-i)}.$$

In order to find the elements of the type $u_{0,k}$, the 0 degree in z of these equations should be extracted.

Knowing that

$$\forall z \in]-1; +1[, (1+z)^{-\alpha} = 1 + \sum_{n=1}^{+\infty} (-1)^n C_{\alpha+n-1}^n z^n, \quad \text{(C.7)}$$

equating the 0 degrees in z of both sides of equation (C.5) gives

$$u_{0,2k} = \rho'^{k-1} \sum_{i=0}^{k-1} (-1)^i C_{2k-i-3}^{k-2} u_{i,2} - \sum_{i=1}^{k-1} \rho'^i C_{2i-1}^i u_{0,2(k-i)} + \sum_{i=0}^{k-2} \rho'^i \left(\sum_{j=0}^i (-1)^j \frac{\lambda'^j}{j!} C_{2i-j}^i \right) u_{0,2(k-i)-1} \quad (\text{C.8})$$

and the same for equation (C.6):

$$u_{0,2k+1} = \rho'^k \sum_{i=0}^k (-1)^i C_{2k-i-1}^{k-1} u_{i,1} - \sum_{i=1}^k \rho'^i C_{2i-1}^i u_{0,2(k-i)+1} + \sum_{i=0}^{k-1} \rho'^i \left(\sum_{j=0}^i (-1)^j \frac{\lambda'^j}{j!} C_{2i-j}^i \right) u_{0,2(k-i)}. \quad (\text{C.9})$$

Now, the system of equations (C.8) and (C.9) gives an expression of $u_{0,k}$ in function of $u_{0,i}$ for $i < k$. In the next section, we solve this recurrence by searching the z -transform $U_0(z)$ of $u_{0,k}$ and making the link between $U_0(z)$ and $M_1(z)$.

Appendix C.2. Second step - finding $M_1(z)$

Let us define the following z -transforms.

$$U_0^{(e)}(z) = \sum_{k=1}^{+\infty} u_{0,2k} z^{2k} \quad (\text{C.10})$$

$$U_0^{(o)}(z) = \sum_{k=0}^{+\infty} u_{0,2k+1} z^{2k+1} \quad (\text{C.11})$$

$$U_0(z) = U_0^{(e)}(z) + U_0^{(o)}(z) \quad (\text{C.12})$$

$$\forall i \geq 0, U_i(z) = \sum_{k=0}^{+\infty} u_{i,k} z^k \quad (\text{C.13})$$

Then, using notations (B.1), (B.2), (B.3) and (B.4), the system of equations (C.8) and (C.9) becomes

$$\begin{cases} u_{0,2k} = \sum_{i=0}^{k-1} a_{2k,i} u_{i,2} + \sum_{i=0}^{k-2} b_i u_{0,2(k-i)-1} - \sum_{i=1}^{k-1} c_i u_{0,2(k-i)} \end{cases} \quad (\text{C.14})$$

$$\begin{cases} u_{0,2k+1} = \sum_{i=0}^k a_{2k+1,i} u_{i,1} + \sum_{i=0}^{k-1} b_i u_{0,2(k-i)} - \sum_{i=1}^k c_i u_{0,2(k-i)+1} \end{cases} \quad (\text{C.15})$$

Multiplying equation (C.14) by z^{2k} and summing for $k \geq 2$, multiplying equation (C.15) by z^{2k+1} and summing for $k \geq 1$ and summing both leads to

$$\begin{aligned} \sum_{k \geq 3}^{+\infty} u_{0,k} z^k = & \sum_{i=0}^{+\infty} \left[u_{i,1} A_i^{(o)}(z) + u_{i,2} A_i^{(e)}(z) \right] - u_{0,1} z \sum_{i=0}^{+\infty} b_i z^{2i+1} + \left[U_0^{(e)}(z) + U_0^{(o)}(z) \right] \left[\sum_{i=0}^{+\infty} b_i z^{2i+1} - \sum_{i=1}^{+\infty} c_i z^{2i} \right] \end{aligned} \quad (\text{C.16})$$

or, equivalently,

$$\begin{aligned} U_0(z) & \left[1 - b_0 z + \sum_{i=1}^{+\infty} (c_i z^{2i} - b_i z^{2i+1}) \right] \\ & = \sum_{i=0}^{+\infty} \left[u_{i,1} A_i^{(o)}(z) + u_{i,2} A_i^{(e)}(z) \right] + u_{0,1} z + u_{0,2} z^2 - u_{0,1} z \sum_{i=0}^{+\infty} b_i z^{2i+1} \end{aligned} \quad (\text{C.17})$$

Moreover, multiplying equations (C.1) by z^k for every k and summing for all $k \geq 3$, leads to

$$U_0(z) - u_{0,2} z^2 - u_{0,1} z = z [U_0(z) - u_{0,1} z] - \rho' z^2 U_1(z) \quad (\text{C.18})$$

and, since $\mathfrak{M}_{1,k} = u_{1,k} \times \frac{1!}{\lambda^{\frac{1}{\lambda}}}$,

$$M_1(z) = \frac{1}{\lambda'} U_1(z). \quad (\text{C.19})$$

Combining (C.18) and (C.19),

$$M_1(z) = \frac{u_{0,2} - u_{0,1}}{\rho} + \frac{u_{0,1}}{\rho} \frac{1}{z} + \frac{z-1}{\rho z^2} U_0(z) \quad (\text{C.20})$$

Finally, thanks to (C.17),

$$\begin{aligned} M_1(z) & = \frac{1}{\rho z^2 \left[1 - b_0 z + \sum_{i=1}^{+\infty} (c_i z^{2i} - b_i z^{2i+1}) \right]} \times \\ & \left[(z-1) \sum_{i=0}^{+\infty} (u_{i,1} A_i^{(o)}(z) + u_{i,2} A_i^{(e)}(z)) \right. \\ & \left. + (u_{0,1} z + u_{0,2} z^2 - u_{0,1} z^2) \sum_{i=1}^{+\infty} c_i z^{2i} - u_{0,2} z^2 \sum_{i=1}^{+\infty} b_i z^{2i+1} \right] \end{aligned} \quad (\text{C.21})$$

Replacing $u_{\alpha,k}$ by $\mathfrak{M}_{\alpha,k}$, we get (B.12).

At last, by integrating by parts, $\mathfrak{M}_{\alpha,1} = 1 - \frac{\rho}{\lambda'} \mathfrak{M}_{\alpha-1,1}$ and $\mathfrak{M}_{\alpha,2} = 1 - \rho' - \frac{\rho}{\alpha+1} - \frac{\rho}{\lambda'} \mathfrak{M}_{\alpha-1,2}$. Then, by recurrence (B.13) and (B.14) are obtained.

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